

Sum Uncertainty Relation in Quantum Theory

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We prove a new sum uncertainty relation in quantum theory which states that the uncertainty in the sum of two or more observables is always less than or equal to the sum of the uncertainties in corresponding observables. This shows that the quantum mechanical uncertainty in any observable is a convex function. We prove that if we have a finite number N of identically prepared quantum systems, then a joint measurement of any observable gives an error \sqrt{N} less than that of the individual measurements. This has application in quantum metrology that aims to give better precision in the parameter estimation. Furthermore, this proves that a quantum system evolves slowly under the action of a sum Hamiltonian than the sum of individuals, even if they are non-commuting.

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Unlike in classical physics, there are restrictions in quantum theory on how accurately one can measure observables even in principle [1]. This is well documented by the famous Heisenberg uncertainty relation for position and momentum of a quantum particle [2]. Later on the uncertainty relation was generalized for any two non-commuting observables [3]. It tells that the product of uncertainties in two non-commuting observables in a given quantum states is greater than or equal to the average of their commutator in the corresponding quantum state. The Heisenberg uncertainty relation is then a special case of this generalized uncertainty relation.

Here we ask, given two or more observables of a quantum system if one measures their sum, will the uncertainty be more or less than the sum of their individual uncertainties? It turns out that the error introduced in the sum of observables is always less than or equal to the sum of the errors introduced by individual observables. This we term as the sum uncertainty relation—which forms the basis to show that quantum mechanical uncertainty in any observable is actually a convex function. Furthermore, we show that if we have a finite number of identically prepared quantum systems, then the measurement of the collective observable gives an error which is \sqrt{N} smaller than the one obtained via individual measurements. We apply these ideas in quantum metrology that aims to give better precision in the parameter estimation. Moreover, we will give some examples and illustrate the relation for some simple quantum mechanical systems. One consequence of the sum uncertainty relation is that a quantum system evolves more slowly under the action of a sum Hamiltonian than the sum of either separately, i.e., mixing of even non-commuting Hamiltonians slows down the system.

Sum uncertainty relation: Consider a quantum state $|\Psi\rangle$ in a Hilbert space \mathcal{H} . Let A and B are two general observables (they could be commuting or non-commuting) that represent some physical quantities. Then, the quantum mechanical uncertainties associated with these observables in the state $|\Psi\rangle$ are defined via $\Delta A^2 = \langle \Psi | A^2 | \Psi \rangle - \langle \Psi | A | \Psi \rangle^2$ and $\Delta B^2 = \langle \Psi | B^2 | \Psi \rangle - \langle \Psi | B | \Psi \rangle^2$. Similarly, we can define the uncertainty in the sum of two observables as $\Delta(A+B)^2 = \langle \Psi | (A+B)^2 | \Psi \rangle - \langle \Psi | (A+B) | \Psi \rangle^2$. Here, we address the question: what is the relation between $\Delta(A+B)$,

ΔA , and ΔB ? The following theorem answers this.

Theorem: Quantum fluctuation in the sum of *any* two observables is always less than or equal to the sum of their individual fluctuations, i.e., $\Delta(A+B) \leq \Delta A + \Delta B$.

Proof: Let A and B are two observables which could be commuting or non-commuting. Let us define two unnormalized vectors $|\Psi_1\rangle = (A - \langle A \rangle)|\Psi\rangle$ and $|\Psi_2\rangle = (B - \langle B \rangle)|\Psi\rangle$, where $\langle A \rangle = \langle \Psi | A | \Psi \rangle$ and $\langle B \rangle = \langle \Psi | B | \Psi \rangle$ are quantum mechanical averages of A and B , respectively in the state $|\Psi\rangle$. Consider the norm of sum of two vectors $|\Psi_1\rangle + |\Psi_2\rangle$. This is given by

$$\begin{aligned} \|\Psi_1 + \Psi_2\|^2 &= \|\Psi_1\|^2 + \|\Psi_2\|^2 + 2\text{Re}\langle \Psi_1 | \Psi_2 \rangle \\ &= \Delta A^2 + \Delta B^2 + 2\text{Re}\langle \Psi_1 | \Psi_2 \rangle, \end{aligned} \quad (1)$$

where $\|\Psi_1\|^2 = \langle \Psi_1 | \Psi_1 \rangle = \langle \Psi | (A - \langle A \rangle)^2 | \Psi \rangle = \Delta A^2$ and $\|\Psi_2\|^2 = \langle \Psi_2 | \Psi_2 \rangle = \langle \Psi | (B - \langle B \rangle)^2 | \Psi \rangle = \Delta B^2$. Using the fact that $\text{Re}\langle \Psi_1 | \Psi_2 \rangle \leq |\langle \Psi_1 | \Psi_2 \rangle|$ and further using the Schwartz inequality we have $\text{Re}\langle \Psi_1 | \Psi_2 \rangle \leq \|\Psi_1\| \|\Psi_2\|$. Then the norm of sum of two vectors satisfy

$$\|\Psi_1 + \Psi_2\|^2 \leq \Delta A^2 + \Delta B^2 + 2\Delta A \Delta B. \quad (2)$$

On the other hand direct evaluation of $\|\Psi_1 + \Psi_2\|^2$ gives

$$\begin{aligned} \|\Psi_1 + \Psi_2\|^2 &= \langle \Psi | [(A - \langle A \rangle) + (B - \langle B \rangle)]^2 | \Psi \rangle \\ &= \langle \Psi | (A+B)^2 | \Psi \rangle - \langle \Psi | (A+B) | \Psi \rangle^2 \\ &= \Delta(A+B)^2. \end{aligned} \quad (3)$$

Thus, (2) and (3) imply that

$$\Delta(A+B) \leq \Delta A + \Delta B \quad (4)$$

which is the sum uncertainty relation. Hence, the proof.

The physical meaning of the sum uncertainty relation is that if we have an ensemble of quantum systems then the ignorance in totality is always less than the sum of the individual ignorance. In case of two observables, if we prepare a large number of quantum systems in the state $|\Psi\rangle$, and then perform the measurement of A on some of those systems and B on some others, then the standard deviations in A plus B will

be more than the standard deviation in the measurement of $(A + B)$ on those systems. Hence, it is always advisable to go for ‘total measurement’ if we want to minimize the error.

In fact, it is not difficult to see that if we have more than two observables (say three observables A , B , and C), then the sum uncertainty relation will read as

$$\Delta(A + B + C) \leq \Delta A + \Delta B + \Delta C. \quad (5)$$

In general for N observables A_1, A_2, \dots, A_N , we will have the sum uncertainty relation as

$$\Delta\left(\sum_i A_i\right) \leq \sum_i \Delta A_i, (i = 1, 2, \dots, N). \quad (6)$$

Convexity of quantum uncertainty: The above inequality brings out an important property of the quantum uncertainty with convexity of a function. To be specific, we will show that the quantum mechanical uncertainty in any observable is actually a convex function. Recall that f is a convex function if

$$f\left(\sum_i p_i x_i\right) \leq \sum_i p_i f(x_i), (i = 1, 2, \dots, N) \quad (7)$$

where p_i ’s satisfy $0 < p_i < 1$, $\sum_i p_i = 1$ and x_i is in the set S [4]. Note that under scaling transformation of an operator $A \rightarrow \lambda A$, the quantum mechanical uncertainty transforms as $\Delta A \rightarrow \lambda \Delta A$. Using this fact we can have a more general sum uncertainty relation. For example, if we have a sum observable $\sum_i p_i A_i$, with all p_i ’s as positive numbers, then we have the following sum uncertainty relation

$$\Delta\left(\sum_i p_i A_i\right) \leq \sum_i p_i \Delta A_i, (i = 1, 2, \dots, N). \quad (8)$$

The meaning of this general sum uncertainty relation is that ‘mixing of commuting or non-commuting operators’ always decreases the uncertainty.

Now, note that in (8) if the positive numbers p_i ’s satisfy $0 < p_i < 1$ and $\sum_i p_i = 1$, then Δ is indeed a convex function. Furthermore, it is known that if f_1, f_2, \dots, f_n are convex functions on \mathbf{R} and $p_i \geq 0$, $(i = 1, 2, \dots, n)$, then $f(x) = \sum_i p_i f_i(x)$ is also a convex function on \mathbf{R} . By using this property of convex function, we can draw the following conclusion. Suppose we have several quantum states $|\Psi_1\rangle, |\Psi_2\rangle, \dots, |\Psi_n\rangle$ and the quantum mechanical uncertainties in A , in the above states are $\Delta_1, \Delta_2, \dots, \Delta_n$. Then, it follows that $\Delta = \sum_i p_i \Delta_i$ is a convex function. The usefulness of convexity of quantum uncertainty may be similar to the entropy of a quantum system. It may be mentioned that entropy of a quantum state is a concave function. Also, we know that entropy signifies the information content of a quantum state. So, in that sense one can think of negative of quantum uncertainty as a concave function and it may represent as a ‘measure of information’ (ignorance). Thus, this property unravels another feature of quantum mechanical uncertainty.

Error in collective and individual measurements: We can test the inequality (6) explicitly by considering an ensemble that

consists of N -identically prepared quantum systems. Let each system be in the state $|\Psi\rangle$. Therefore, the combined state vector of N -particle is given by

$$|\Psi\rangle^{\otimes N} = |\Psi\rangle_1 \otimes |\Psi\rangle_2 \otimes \dots \otimes |\Psi\rangle_N. \quad (9)$$

Let us first measure an observable A on each particle individually (not collectively). The individual observables of interest are $A_1 = A \otimes I \otimes \dots \otimes I$, $A_2 = I \otimes A \otimes \dots \otimes I$, ..., and $A_N = I \otimes I \otimes \dots \otimes A$. Then, one can check that the average of A_i ($i = 1, 2, \dots, N$) in the state $|\Psi\rangle^{\otimes N} = \langle\Psi|A|\Psi\rangle$. Similarly, the uncertainty in each A_i is ΔA . Therefore, the sum of uncertainties in the individual measurements is $\sum_i \Delta A_i = N \Delta A$.

Now, suppose we perform measurement of the sum observable on N -copies. The sum observable is given by

$$A_S = \sum_i A_i = A \otimes I \otimes \dots \otimes I + I \otimes A \otimes \dots \otimes I + \dots + I \otimes I \otimes \dots \otimes A. \quad (10)$$

The quantum mechanical uncertainty in the observable A_S in the state $|\Psi\rangle^{\otimes N}$ is given by

$$\Delta A_S^2 = {}^{\otimes N}\langle\Psi|A_S^2|\Psi\rangle^{\otimes N} - ({}^{\otimes N}\langle\Psi|A_S|\Psi\rangle^{\otimes N})^2. \quad (11)$$

Note that ${}^{\otimes N}\langle\Psi|A_S^2|\Psi\rangle^{\otimes N} = N\langle\Psi|A^2|\Psi\rangle + 2N\langle\Psi|A|\Psi\rangle^2$ and ${}^{\otimes N}\langle\Psi|A_S|\Psi\rangle^{\otimes N} = N\langle\Psi|A|\Psi\rangle$. Hence, $\Delta A_S^2 = N\Delta A^2$. This implies that the quantum mechanical uncertainty in the sum observable is $\Delta A_S = \sqrt{N}\Delta A$. Therefore, the sum uncertainty relation reads as

$$\sqrt{N}\Delta A \leq N\Delta A \quad (12)$$

which is clearly satisfied. This analysis also suggests that the error in the total measurement goes as $\sqrt{N}\Delta A$, whereas the sum of errors in the individual measurement goes as $N\Delta A$. Thus, there is an overall \sqrt{N} improvement in the error of measurement of sum observable with N -copies.

Parameter estimation and quantum metrology: Precision measurement which requires estimation of some parameter to its highest accuracy is an important problem. If one uses laws of quantum theory, then in the measurement of some parameter the precision can be enhanced. This a topic of great study in quantum metrology. In this scheme one prepares a probe state $|\psi_0\rangle$, applies a unitary operator $U(\theta)$ that depends on the parameter θ to be estimated and then measures some observable X on the resulting state $|\psi(\theta)\rangle$. If $U(\theta) = \exp(-i\theta H)$ where H is a Hermitian operator, then using the Mandelstam-Tamm uncertainty relation [5, 6], we have

$$\Delta X \Delta H \geq \frac{1}{2} |\partial\langle X \rangle / \partial\theta|. \quad (13)$$

The precision with which one can estimate θ is given by

$$\delta\theta = \Delta X / |\partial\langle X \rangle / \partial\theta| \geq \frac{1}{2\Delta H}, \quad (14)$$

where ΔX , ΔH , and $\langle X \rangle$ have their usual meaning in the quantum state $|\psi(\theta)\rangle$. Therefore, if we want to minimize the

error in estimating the parameter, we have to minimize ΔX or maximize ΔH . How to achieve that goal is the subject of quantum metrology [7]. It turns out that using quantum entangled probe states or entangling unitary operator one can achieve better and better precision in the parameter estimation.

Recently, Giovannetti *et al* [8] have shown that that using entangled probe state one can achieve an enhancement that scales as $1/N$. More recently, it was shown by Roy and Braunstein [9] that if one exploits entangling unitary operator then one obtains an exponential enhancement in the parameter estimation. In particular, by choosing an appropriate Hamiltonian one can apply the unitary operator $U = e^{-i\theta H}$ and generate a N -qubit state given by (for details see [9])

$$\begin{aligned} |\psi_H(\theta)\rangle &= e^{-i\theta H} |00 \dots 00\rangle \\ &= \cos(2^{N-1}\theta) |00 \dots 00\rangle - i \sin(2^{N-1}\theta) |11 \dots 11\rangle. \end{aligned} \quad (15)$$

Then, by measuring the observable $X = \otimes_{i=1}^N P_j$, where $P_j = |0\rangle_j \langle 0|$ one can estimate θ as given by $\delta\theta = 1/2^N$ which is the exponential enhancement in the precision.

Here, we show that there are other class of measurement strategies also which can give the same precision. Suppose, instead of measuring the product observable we measure the sum observable, i.e., measure $X = \sum_i P_i = P_S = P_1 \otimes I \otimes \dots \otimes I + I \otimes P_2 \otimes \dots \otimes I + \dots + I \otimes I \otimes \dots \otimes P_N$. Then, the precision in the parameter estimation is given by

$$\delta\theta = \Delta P_S / |\partial \langle P_S \rangle / \partial \theta|. \quad (16)$$

The quantum uncertainty and average for P_S in the state $|\psi_H(\theta)\rangle$ are given by

$$\Delta P_S = \frac{N}{2} \sin(2^N \theta), \quad \langle P_S \rangle = N \cos^2(2^{N-1} \theta). \quad (17)$$

Therefore, the precision in the estimation of the parameter θ is $1/2^N$. Similarly, if we measure individually these projectors then the corresponding precision in the parameter θ will be

$$\delta\theta = \sum_i \Delta P_i / |\partial (\sum_i \langle P_i \rangle) / \partial \theta|. \quad (18)$$

One can check that for the state $|\psi_H(\theta)\rangle$, we have $\sum_i \Delta P_i = \frac{N}{2} \sin(2^N \theta)$ and $\sum_i \langle P_i \rangle = N \cos^2(2^{N-1} \theta)$. Thus, again we see that $\delta\theta = 1/2^N$. Hence, for the entangling unitary, joint measurement and individual measurements give the same precision as obtained in [9]. The physical explanation is now clear. This is happening because, the sum uncertainty relation is saturated for these observables and the equality holds.

One may wonder if by increasing the resources and by exploiting the sum uncertainty relation (i.e. the idea that the measurement of sum observable minimizes the error leading to better precision) one can enhance the precision in the parameter better than the exponential [9]. However, as we will see this is not the case. Let us imagine that we have an ensemble of N -probe states. The number of copies of N -probe states are finite (say) M . On each of the N -probe state we apply the entangling unitary operator as suggested by Roy and

Braunstein [9]. But there is no further interaction between these M -copies. Then, the combined state of MN probe state is given by

$$|\psi_H(\theta)\rangle^{\otimes M} = |\psi_H(\theta)\rangle_1 \otimes |\psi_H(\theta)\rangle_2 \otimes \dots \otimes |\psi_H(\theta)\rangle_M. \quad (19)$$

On these collection of MN probe states we measure a sum observable. The observable of interest is

$$\begin{aligned} \Pi_S &= \Pi_1 \otimes I_2 \otimes \dots \otimes I_M + I_1 \otimes \Pi_2 \otimes \dots \otimes I_M + \dots \\ &+ I_1 \otimes I_2 \otimes \dots \otimes \Pi_M, \end{aligned} \quad (20)$$

where Π_j , ($j = 1, 2, \dots, M$) are product of projection operators on j th copy of the N -probe state. To be clear, $\Pi_1 = P = \otimes_{i=1}^N P_i$ on the 1st N -probe state, $\Pi_2 = P = \otimes_{i=1}^N P_i$ on the 2nd N -probe state and so on. The precision in the measurement of the sum observable Π_S is given by

$$\delta\theta = \Delta \Pi_S / |\partial \langle \Pi_S \rangle / \partial \theta|. \quad (21)$$

One can check that the quantum uncertainty and average in Π_S for the MN probe state $|\psi_H(\theta)\rangle^{\otimes M}$ are given by

$$\Delta \Pi_S = \frac{\sqrt{M}}{2} \sin(2^N \theta), \quad \langle \Pi_S \rangle = M \cos^2(2^{N-1} \theta). \quad (22)$$

Therefore, the precision in the estimation of the parameter θ is given by

$$\delta\theta = 1/\sqrt{M} 2^N. \quad (23)$$

This result apparently may give an impression that this is better than exponential [9]. But, if we re-express the result in terms of actual resource used, i.e., the number $K = MN$, then the precision $\delta\theta = \sqrt{N} 2^{K(\frac{M-1}{M})} \frac{1}{\sqrt{K} 2^K}$, which is lower than the exponential. Therefore, it is always not the case that by using more resources one can enhance the precision.

Now, we give few further applications of the sum uncertainty relation in quantum theory. First, we apply to Hamiltonian systems and second we apply to the speed of quantum mechanical systems.

Uncertainty in the Hamiltonian: One immediate application is that for any quantum mechanical system, the total Hamiltonian H consists of kinetic and potential energy, i.e., $H = T + V$ and using the sum uncertainty relation we have $\Delta H \leq \Delta T + \Delta V$. Thus, the uncertainty in the total energy in any state is bounded by the sum of uncertainties in the kinetic and potential energy. This result is interesting, in the sense that if we want to do energy measurement with minimal error, then do not measure kinetic and potential energy separately. Always measure the total energy because the quantum mechanical uncertainty is less in that case. Also, this shows that the uncertainty in the total energy is less than the uncertainty in the kinetic energy in the position basis. However in the momentum basis, the uncertainty in the total energy is less than the uncertainty in the potential energy. These observations may have many applications in the complex quantum systems.

Sub-additivity of quantum speed: Here, we ask whether the speed of evolution of a state vector through Hilbert space behaves like the classical speed. In what follows, we will show that the speed with which a quantum system evolves under two Hamiltonians (commuting or non-commuting) are not added up. (Note that classically, if a particle is subjected to two force fields, then the speed of a particle is added up.)

In quantum theory, when a system evolves under some Hamiltonian H , then the state undergoes a continuous time evolution, i.e., $|\Psi(0)\rangle \rightarrow |\Psi(t)\rangle = \exp(-iHt)|\Psi(0)\rangle$. One can ask how fast does the system evolve in time. Then, the rate at which it evolves is nothing but the speed of transportation of the state vector in the projective Hilbert space [10, 11]. This is defined as $v = \frac{dD}{dt}$, where dD is the infinitesimal distance between nearby quantum states $|\Psi(t)\rangle$ and $|\Psi(t+dt)\rangle$. The distance function is given by

$$dD^2 = (1 - |\langle\Psi(t)|\Psi(t+dt)\rangle|^2) = \frac{dt^2}{\hbar^2} \Delta H^2, \quad (24)$$

where ΔH is the usual uncertainty in the Hamiltonian in the state $|\Psi\rangle$. Therefore, the speed at which a quantum system evolves is nothing but the uncertainty in the Hamiltonian of the system, i.e., $v = \Delta H/\hbar$. This is the geometric meaning of quantum fluctuation: more the fluctuation in the Hamiltonian, faster the system will evolve.

Now, imagine that a quantum system evolves under a Hamiltonian H_1 , then the speed is given by $v_1 = \Delta H_1/\hbar$. Similarly, if this evolves under a Hamiltonian H_2 , then the speed is given by $v_2 = \Delta H_2/\hbar$. Suppose, now the system evolves under the Hamiltonian $H = H_1 + H_2$. What will be the speed? Will the total speed be $v = v_1 + v_2$? The answer is no. Using the sum uncertainty relation we see that $v = \Delta H/\hbar \leq \Delta H_1/\hbar + \Delta H_2/\hbar$. In other words, the quantum speed obeys the relation

$$v \leq v_1 + v_2. \quad (25)$$

The meaning of this equation is that in general a quantum system will evolve more slowly under the action of a sum Hamiltonian than the sum of either separately. This is a non-trivial result, in the sense that this holds for generic Hamiltonians be they commuting or non-commuting. This is something counter intuitive which arise due to quantum mechanical nature of the associated observables and also the fact that quantum systems obey the Schrödinger equation and not the Newton equation. Also, it may be noted that if we have a Hamiltonian $H = H_1 - H_2$, the speed will obey the relation

$$v \leq v_1 + v_2. \quad (26)$$

This is due to the fact that $\Delta(-A) = \Delta A$, i.e., quantum mechanical uncertainty is an even function.

One may ask whether the velocity operator of a quantum system obeys the sub-additivity condition. We will show that in general this may not. But the average of the velocity operator may obey a kind of sub-additivity condition. Note that using the Heisenberg equation of motion, the velocity operator can be defined as $v = \frac{1}{i\hbar}[x, H]$, where x is the position

and H is the Hamiltonian. Then, the magnitude of the average of the velocity operator will obey $|\langle v \rangle| \leq \frac{2}{\hbar} \Delta x \Delta H$. Now, if we have a Hamiltonian H_1 , then the velocity operator will obey $|\langle v_1 \rangle| \leq \frac{2}{\hbar} \Delta x \Delta H_1$. Similarly, for a Hamiltonian H_2 , the velocity $|\langle v_2 \rangle| \leq \frac{2}{\hbar} \Delta x \Delta H_2$. This implies that $|\langle v_1 \rangle|_{\max} = \frac{2}{\hbar} \Delta x \Delta H_1$ and $|\langle v_2 \rangle|_{\max} = \frac{2}{\hbar} \Delta x \Delta H_2$. Now, using the sum uncertainty relation we have

$$|\langle v \rangle| \leq \frac{2}{\hbar} \Delta x (\Delta H_1 + \Delta H_2). \quad (27)$$

This suggests that $|\langle v \rangle| \leq |\langle v_1 \rangle|_{\max} + |\langle v_2 \rangle|_{\max}$. This is another interesting application of the sum uncertainty relation.

Conclusion: We have proved a new sum uncertainty relation for general observables in quantum theory which shows that quantum mechanical uncertainty in the sum of two or more observables is always less than or equal to the sum of quantum uncertainties in the individual observables. We have also proved that the quantum mechanical uncertainty is indeed a convex function. This property suggests that there is some analogy between quantum uncertainty and entropy of a quantum mechanical system. We have shown that if we have a finite number of identically prepared quantum states, then there is an overall \sqrt{N} improvement in the error of measurement of the sum observable with N -copies. As an important application we have explained why the measurement of the sum and individual observables can give the same exponential precision. Also, we have shown that using more resources one cannot have a precision better than the exponential one. In addition, we prove that in general a quantum system evolves more slowly under the action of a sum Hamiltonian than the sum of either separately. It is expected that the sum uncertainty relation will have wider applications in a variety of context like quantum computation, quantum information theory and many body quantum systems.

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